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LETTER TO THE EDITOR

Polynomial solutions to the WDVV equations in four dimensions

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Abstract. All polynomial solutions of the WDVV equations for the case $n = 4$ are determined. We find all five solutions predicted by Dubrovin, namely those corresponding to Frobenius structures on orbit spaces of finite Coxeter groups. Moreover we find two additional series of polynomial solutions of which one series is of semi-simple type (massive). This result supports Dubrovin's conjecture if modified appropriately.

1. Introduction

Recently in the physics literature on two-dimensional topological field theory [1, 3] a remarkably and amazingly rich system of partial differential equations appeared. Roughly speaking, this system describes the conditions for a quasi-homogeneous function $F = F(t)$ of the variable $t = (t_1, \ldots, t_n)$ such that the third-order derivatives form the structure constants of an associative algebra. This system of equations is known as the Witten–Dijkgraaf–H Verlinde–E Verlinde (WDVV) system.

In the paper [2] and the review article [3], Dubrovin describes, given any finite Coxeter group, how to determine a polynomial solution of the WDVV system. Furthermore, he shows that the algebras associated to these polynomial solutions satisfy a certain semi-simplicity assumption. More generally Dubrovin conjectures that any polynomial solution of the WDVV system with positive degrees, such that the associated algebra is semi-simple (massive), can be obtained in such a way.

In this letter we discuss the case $n = 4$. We have determined all polynomial solutions with positive degrees, see the appendix. We recover the five solutions associated to the Coxeter groups A_4, B_4, D_4, F_4 and H_4. However, we also find two series of additional polynomial solutions. Only one of these series yields semi-simple algebras. This series corresponds to the direct product of two (irreducible) Coxeter groups. This result supports Dubrovin's conjecture if modified slightly, such that appropriate reducible Coxeter groups are included.

2. The WDVV equations

2.1. Definition

Our aim is to find functions $F(t) = F(t^1, \ldots, t^n)$ such that the third-order derivatives,

$$c_{\alpha\beta\gamma}(t) = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial^\gamma}$$

obey the following conditions, cf [3].

1. *Normalization*

$$c_{1\alpha\beta} = \begin{cases} 0 & \text{if } \alpha+\beta \neq n+1 \\ 1 & \text{if } \alpha+\beta = n+1. \end{cases}$$

We introduce the metric $\eta^{\alpha\beta} = \eta_{\alpha\beta} = c_{1\alpha\beta}$.

2. *Associativity*

The functions

$$c^{\gamma}_{\alpha\beta}(t) = \sum_{\epsilon} \eta^{\gamma\epsilon} c_{\epsilon\alpha\beta}$$

for any t, must define an n-dimensional associative algebra with basis $e_1, \ldots, e_n$ and product given by

$$e_\alpha \cdot e_\beta = \sum_{\gamma} c^{\gamma}_{\alpha\beta} e_\gamma .$$

3. *Homogeneity*

$F(t)$ must be quasihomogeneous in the variables $t^1, \ldots, t^n$, i.e. there must exist constants $d_1, d_2, \ldots, d_n$ and d_F such that

$$\sum_{\alpha} d_\alpha t^\alpha \frac{\partial F}{\partial t^\alpha} = d_F F. \tag{2.1}$$

This system of conditions, we call, following [3], the WDVV equations [1, 4]. We assumed that the metric $\eta_{\alpha\beta}$ is in standard form. We will restrict our attention to the case that $d_1, \ldots, d_n$ are all *strictly positive*. In this case we will assume (unless stated differently) that $d_1 = 1$, and we will write, following physical conventions, $d_F = 3 - d$. For physical reasons we will also assume that $d > 0$.

The associativity condition leads to an overdetermined system of partial differential equations (PDEs),

$$\sum_{\lambda} \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial^\lambda} \cdot \frac{\partial^3 F(t)}{\partial t^\gamma \partial t^\delta \partial^{n+1-\lambda}} = \sum_{\lambda} \frac{\partial^3 F(t)}{\partial t^\gamma \partial t^\beta \partial^\lambda} \cdot \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\delta \partial^{n+1-\lambda}} \tag{2.2}$$

for any α, β, γ and δ. From these equations, the dependency of F on t^1 can be solved completely. Namely we get

$$F(t) = \frac{1}{2} t^1 \sum_{\alpha=2}^{n} t^\alpha t^{n-\alpha+1} + f(t^2, \ldots, t^n). \tag{2.3}$$

Now for F to be homogeneous we need (assuming $d_1 = 1$ and $d_F = 3 - d$) that

$$d_n = 1 - d \qquad \text{and} \qquad d_\alpha + d_{n+1-\alpha} = 2 - d. \tag{2.4}$$

Since we require $d_n > 0$, we need that $d < 1$. Hence $0 < d < 1$.

3. Polynomial solutions for $n = 4$

3.1. The results

We will denote t^2, t^3, t^4 by x, y, z. The system (2.2), taking into account (2.3), is now equivalent to

$$f_{zzz} = f_{xyz}^2 - f_{xxy} f_{yzz} - f_{xyy} f_{xzz} + f_{xxz} f_{yyz} \tag{3.1}$$

$$f_{xzz} = -f_{xyy} f_{xxz} + f_{xxx} f_{yyz} \tag{3.2}$$

$$f_{yzz} = -f_{xxy} f_{yyz} + f_{yyy} f_{xxz} \tag{3.3}$$

$$f_{xyz} = -\tfrac{1}{2} f_{xyy} f_{xxy} + \tfrac{1}{2} f_{xxx} f_{yyy}. \tag{3.4}$$

We have determined all polynomial solutions corresponding to the case $d_1 = 1$, $d_2 \geqslant d_3 > 0$, $d_4 > 0$ and $d > 0$. This was done using the computer algebra package REDUCE [5]. The technique that Dubrovin uses for $n = 3$ does not work for $n = 4$. However, one can deduce from the conditions on d_1, d_2, d_3, d_4 and d that f is at most cubic in x. This splits the system (3.1)–(3.4) into a system of 18 PDEs, now in two variables y and z. Carefully analysing this system leads to the seven solutions presented in the appendix.

3.2. Discussion of the results

Studying the seven classes of solutions, we find that the first five correspond to Coxeter groups. For (A.6) it turn out that for generic c, t and q there are no nilpotent elements. For (A.7) the situation is completely different. In this case we have that $e_2^2 = c_{21}(2e_2 - c_{21}e_1)$, independent of t and q. Now taking $a = c_{21}e_1 - e_2$, we have that $a^2 = 0$. Hence we disregard (A.7).

The solution (A.6) can be interpreted in the following way. In the case $n = 2$ there is one series of polynomial solutions

$$F(t^1, t^2) = \tfrac{1}{2}(t^1)^2 t^2 + \alpha (t^2)^k.$$

Now we can take the sum $F(t^1, t^2) + F(t^3, t^4)$ (with two different α). This will be a solution of the associativity condition; this construction corresponds to taking the direct sum of the algebras, cf [3]. The unit will now correspond to $t^1 + t^3$. Since d_2 and d_4 are the same, F will again be quasi-homogeneous.

Concluding, we can say that in the case $n = 4$ all semi-simple solutions correspond to (irreducible) Coxeter groups, or to direct products of compatible Coxeter groups.

In general, one sees that to direct products of Coxeter groups with the same Coxeter numbers, one can associate solutions of the WDVV system. Hence Dubrovin's conjecture should be modified to include these cases.

Appendix

The weights are rescaled to obtain $d_4 = 2$.

$d = 3$, $d_1 = 5$, $d_2 = 4$, $d_3 = 3$, $d_4 = 2$; corresponds to A_4:

$$f = c_{30}x^3 + 108c_{30}^2c_4x^2z^2 + 36c_{30}c_4xy^2z + c_4y^4 + 864c_{30}^2c_4^2y^2z^3 + \tfrac{93312}{5}c_{30}^4c_4^3z^6 \tag{A.1}$$

$d = 6$, $d_1 = 8$, $d_2 = 6$, $d_3 = 4$, $d_4 = 2$; corresponds to B_4:

$$f = c_{30}x^3 + 9c_{30}c_{13}x^2yz + 54c_{30}^2c_{13}^2x^2z^3 + c_{13}xy^3 + 27c_{30}c_{13}^2xy^2z^2 + \tfrac{9}{4}c_{13}^2y^4z + 27c_{30}c_{13}^3y^3z^3 + \tfrac{1458}{5}c_{30}^2c_{13}^4y^2z^5 + \tfrac{13122}{7}c_{30}^4c_{13}^6z^9 \tag{A.2}$$

$d = 4$, $d_1 = 6$, $d_2 = 4$, $d_3 = 4$, $d_4 = 2$: corresponds to D_4:

$$f = c_{30}x^3z + c_{11}xyz^3 + \tfrac{1}{6}\tfrac{c_{11}}{c_{30}}y^3z + \tfrac{3}{70}c_{11}^2z^7 \tag{A.3}$$

$d = 10$, $d_1 = 12$, $d_2 = 8$, $d_3 = 6$, $d_4 = 2$; corresponds to F_4:

$$f = c_{30}x^3z + \tfrac{36}{5}c_{30}^2c_4x^2z^5 + 12c_{30}c_4xy^2z^3 + c_4y^4z + \tfrac{144}{7}c_{30}^2c_4^2y^2z^7 + \tfrac{1728}{143}c_{30}^4c_4^3z^{13} \tag{A.4}$$

$d = 28$, $d_1 = 30$, $d_2 = 20$, $d_3 = 12$, $d_4 = 2$; corresponds to H_4:

$$f = c_{30}x^3z + \tfrac{9}{5}c_{30}c_{13}x^2yz^5 + \tfrac{72}{275}c_{30}^2c_{13}^2x^2z^{11} + c_{13}xy^3z^3 + \tfrac{3}{5}c_{30}c_{13}^2xy^2z^9 + \tfrac{1}{20}\tfrac{c_{13}}{c_{30}}y^5z + \tfrac{3}{10}c_{13}^2y^4z^7 + \tfrac{3}{25}c_{30}c_{13}^3y^3z^{13} + \tfrac{72}{2375}c_{30}^2c_{13}^4y^2z^{19} + \tfrac{3456}{14046875}c_{30}^4c_{13}^6z^{31} \tag{A.5}$$

$d = q - 3,\ d_1 = q - 1,\ d_2 = q - 1,\ d_3 = 2,\ d_4 = 2;\ q \geqslant 4$:

$$f = (c_{21}y + c_{20}z)x^2 + c_1(y + (-c_{21} + D)z)^q + c_2(y + (-c_{21} - D)z)^q \quad \text{with } D = \sqrt{c_{21}^2 + 2c_{20}} \tag{A.6}$$

$d = 2q - 4,\ d_1 = 2q - 2,\ d_2 = 2q - 2,\ d_3 = 2,\ d_4 = 2;\ q \geqslant 3$:

$$f = c_{21}(y - \tfrac{1}{2}c_{21}z)x^2 + c_{11}(y - c_{21}z)^q x + (c_1 y + c_2 z)(y - c_{21}z)^{2q-2} \tag{A.7}$$

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